New Solitary Wave Solution for the Boussinesq Wave Equation Using the Semi-Inverse Method and the Exp-Function Method

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Using the semi-inverse method, a variational formulation is established for the Boussinesq wave equation. Based on the obtained variational principle, solitary solutions in the sech-function and exp-function forms are obtained.

Key words: Boussinesq Wave Equation; Solitary Wave Solution; Semi-Inverse Method; Exp-Function Method.

1. Introduction

The search for the exact travelling wave solutions to nonlinear disperse wave systems plays an important role in the study of nonlinear physical phenomena. In the past few years, many effective methods have been suggested for searching for soliton solutions for various nonlinear wave equations, among which the variational iteration method [1 – 5], the homotopy perturbation method [6 – 10], the parameter-expansion method [11 – 13], He's variational method [8, 14, 15] and the exp-function method [16 – 20] have been shown to be effective, easy, and accurate for a large class of nonlinear problems. D'Acunto [21, 22] applied He's variational method to the determination of limit cycles. Ozis and Yildirim [23], Zhang [24], and the present author [25] illustrated that He's variational method was also valid for nonlinear wave equations.

In this paper, we will consider the Boussinesq wave equation [26]

$$u_{tt} - u_{xx} + 3(u^2)_{xx} + a u_{xxxx} = 0, \tag{1}$$

which was proposed by Boussinesq to describe surface water waves whose horizontal scale is much larger than the depth of the water. The equation also arises in other physical applications such as nonlinear lattice waves, iron sound waves in a plasma, vibrations in a nonlinear string, and in the percolation of water in porous subsurface strata [27]. Recently, Wazwaz in [26] obtained compactons, solitons, solitary patterns, and periodic solutions of the Boussinesq wave equation (1) by the tanh-method. Javidi and Jalilian in [27] apply the variational iteration method to construct solitary wave solutions of (1).

The aim of this paper is to couple He's semi-inverse method [14] and the exp-function method [16] to search for solitary solutions of the discussed problem.

2. Variational Formulation

According to He's variational method [8] for solitary solutions, a variational formulation should be first established.

For convenience, we seek the travelling wave solution of the Boussinesq wave equation by assuming the solution in the following form:

$$u(x,t) = u(\xi), \quad \xi = x - ct, \tag{2}$$

where c is a constant to be determined.

Substitution the solution (2) into (1) gives

$$(c^2 - 1)u'' + 3(u^2)'' + a u^{(4)} = 0, \tag{3}$$

where the prime expresses the derivative with respect to ξ. Integrating (3) twice, we have

$$(c^2 - 1)u + 3u^2 + a u'' = g_1 \xi + g_2, \tag{4}$$

where g_1 and g_2 are the constants of integration. We set $g_1 = g_2 = 0$ for simplicity, then (4) reduces to

$$(c^2 - 1)u + 3u^2 + a u'' = 0. \tag{5}$$

The variational iteration method and the homotopy perturbation method can be applied to (5) effectively. Here we follow He's variational method described in [8], where the Korteweg-de Vries (KdV) equation was used to illustrate the effectiveness and convenience of the suggested method. According to [8], a variational formulation should be first established using the semi-inverse method [26]:

$$J=\int_0^\infty\left[\frac{1}{2}(c^2-1)u^2+u^3-\frac{1}{2}a\Big(\frac{\mathrm{d}u}{\mathrm{d}\xi}\Big)^2\right]\mathrm{d}\xi. \quad (6)$$

3. Soliton Wave Solution

3.1. The Form of Sech-Function

Following He's variational method suggested in [8], we first search for a solitary solution in the form

$$u=p\,\mathrm{sech}^2(q\xi), \quad (7)$$

where p and q are unknown constants to be further determined. Substituting (7) into (6), we have

$$\begin{aligned}J&=\int_0^\infty\Big[\frac{1}{2}(c^2-1)p^2\,\mathrm{sech}^4(q\xi)+p^3\,\mathrm{sech}^6(q\xi)\\&\qquad-2\alpha p^2q^2\,\mathrm{sech}^4(q\xi)\tanh^2(q\xi)\Big]\mathrm{d}\xi\\&=\frac{(c^2-1)p^2}{3q}+\frac{8p^3}{15q}-\frac{4ap^2q}{15}.\end{aligned} \quad (8)$$

Making J stationary with respect to p and q results in

$$\frac{\partial J}{\partial p}=\frac{2(c^2-1)p}{3q}+\frac{8p^2}{5q}-\frac{8apq}{15}=0, \quad (9)$$

$$\frac{\partial J}{\partial q}=-\frac{(c^2-1)p^2}{3q^2}-\frac{8p^3}{15q^2}-\frac{4ap^2}{15}=0, \quad (10)$$

or simplified

$$5(c^2-1)+12p-4aq^2=0, \quad (11)$$

$$-5(c^2-1)-8p-4aq^2=0. \quad (12)$$

Solving (11) and (12) simultaneously results in

$$p=\frac{1-c^2}{2},\quad q^2=\frac{1-c^2}{4a}. \quad (13)$$

The solitary solution is, therefore, obtained as follows:

$$u_1(x,t)=\frac{1-c^2}{2}\,\mathrm{sech}^2\left[\frac{1}{2}\sqrt{\frac{1-c^2}{a}}(x-ct)\right], \quad (14)$$

for $a>0$ and $c^2<1$. This is exactly the soliton solution obtained in [26].

3.2. The Form of Simple Exp-Function

As mentioned above the exp-function method [16] is very simple and straightforward, which is based on the assumption that travelling wave solutions can be expressed in the following exp-function form:

$$u(\xi)=\frac{\sum_{n=-c}^{d}a_n\exp(n\xi)}{\sum_{m=-p}^{q}b_m\exp(m\eta)},$$

where c, d, p, and q are positive integers which are unknown to be further determined, a_n and b_m are unknown constants. In this paper we use a simplified exp-function method, i. e., we search for a solitary solution in the following simple exp-function form:

$$u=k\exp(-m\xi^2), \quad (15)$$

where k and m are unknown constants to be further determined.

Substituting (15) into (6) and using the integration relations

$$\begin{aligned}\int_0^\infty u^n\mathrm{d}\xi&=k^n\int_0^\infty\exp(-mn\xi^2)\mathrm{d}\xi\\&=\frac{\sqrt{n\pi}}{2n}k^nm^{-\frac{1}{2}},\quad n\in N,\end{aligned} \quad (16)$$

$$\begin{aligned}\frac{1}{2}\int_0^\infty\left(\frac{\mathrm{d}u}{\mathrm{d}\xi}\right)^2\mathrm{d}\xi&=2k^2m^2\int_0^\infty\xi^2\exp(-2m\xi^2)\mathrm{d}\xi\\&=\frac{\sqrt{2\pi}}{8}k^2m^{\frac{1}{2}},\end{aligned} \quad (17)$$

we have

$$\begin{aligned}J&=\frac{1}{2}(c^2-1)\int_0^\infty u^2\mathrm{d}\xi+\int_0^\infty u^3\mathrm{d}\xi\\&\quad-\frac{1}{2}a\int_0^\infty\left(\frac{\mathrm{d}u}{\mathrm{d}\xi}\right)^2\mathrm{d}\xi\\&=\frac{\sqrt{2\pi}}{8}(c^2-1)k^2m^{-\frac{1}{2}}+\frac{\sqrt{3\pi}}{6}k^3m^{-\frac{1}{2}}-\frac{\sqrt{2\pi}}{8}ak^2m^{\frac{1}{2}}.\end{aligned} \quad (18)$$

Making J stationary with respect to k and m results in

$$\begin{aligned}\frac{\partial J}{\partial k}&=\frac{\sqrt{2\pi}}{4}(c^2-1)km^{-\frac{1}{2}}+\frac{\sqrt{3\pi}}{2}k^2m^{-\frac{1}{2}}\\&\quad-\frac{\sqrt{2\pi}}{4}akm^{\frac{1}{2}}=0,\end{aligned} \quad (19)$$

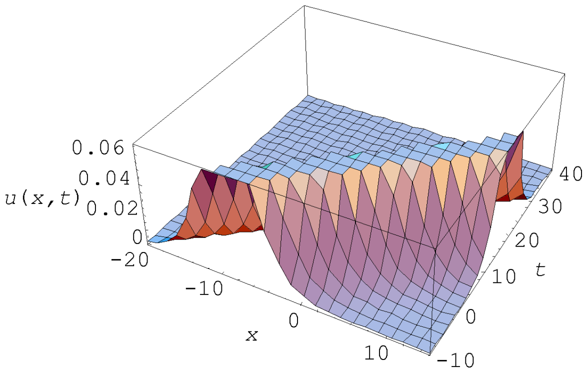

Fig. 1. Solitary wave solution $u_{1,1}(x,t)$.

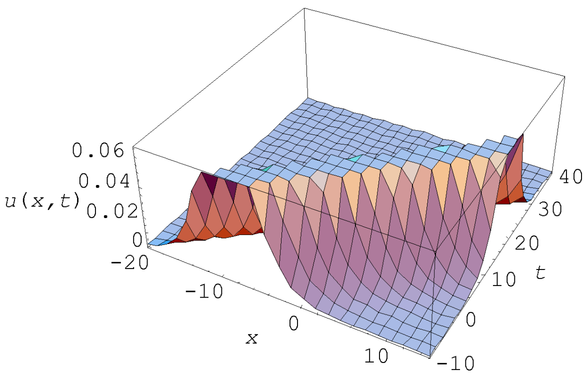

Fig. 2. Solitary wave solution $u_{2,1}(x,t)$.

$$\frac{\partial J}{\partial m} = -\frac{\sqrt{2\pi}}{16}(c^2-1)k^2 m^{-\frac{3}{2}} - \frac{\sqrt{3\pi}}{12}k^3 m^{-\frac{3}{2}} - \frac{\sqrt{2\pi}}{16}ak^2 m^{-\frac{1}{2}} = 0 \tag{20}$$

or simplified

$$\sqrt{2}(c^2-1)+2\sqrt{3}k-\sqrt{2}am=0, \tag{21}$$

$$-3\sqrt{2}(c^2-1)-4\sqrt{3}k-3\sqrt{2}am=0. \tag{22}$$

Solving (21) and (22) simultaneously results in

$$k=\frac{\sqrt{6}(1-c^2)}{5}, \quad m=\frac{1-c^2}{5a}. \tag{23}$$

The solitary solution is, therefore, obtained as follows:

$$u_2(x,t)=\frac{\sqrt{6}(1-c^2)}{5}\exp\left\{-\frac{1-c^2}{5a}(x-ct)^2\right\}. \tag{24}$$

To our knowledge the above form solitary solution is first appeared in literature.

Figures 1 and 2 show the plots of

$$u_{1,1}(x,t)=\frac{1}{8}\operatorname{sech}^2\left[\frac{1}{4}\left(x-\frac{\sqrt{3}}{2}t\right)\right], \tag{25}$$

the solution (14) with $a=1$ and $c=\frac{\sqrt{3}}{2}$, and

$$u_{2,1}(x,t)=\frac{\sqrt{6}}{20}\exp\left\{-\frac{1}{20}\left(x-\frac{\sqrt{3}}{2}t\right)^2\right\}, \tag{26}$$

the solution (24) with $a=1$ and $c=\frac{\sqrt{3}}{2}$.

4. Conclusion

To summarize, we can conclude from the obtained results that the variational approach to the Boussinesq wave equation has been demonstrated successfully. This method is extremely simple in its principle and quite easy to use. We predict the method will become a powerful mathematical tool to find solitary solutions of different forms for various nonlinear equations.

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